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Effect of relaxation and loss parameters on the time-evolution of the gas-laser equation

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1. Introduction

The study of the time-evolution of a gas laser is of great interest to complete the study of a laser. As a result it has attracted the attention of many workers at different times. However, earlier studies on the time-evolution of the photon distribution function have neglected either the nonlinear or the loss terms. Shimoda (1957) in his study of this time variation using a Generating function technique has included the loss terms but his discussions are restricted to linear terms only. On the other hand Lambropolous (1967), Makhov (1962) and few others have confined their discussions to either no loss or low loss

It is felt that the main reason for not considering those simultaneously lies in the difficulty to solve second order non-linear partial differential equation in case of Generating function technique and to deal with the large number of terms in case of Laplace transform method.

In the present paper both those situations are dealt with. The nonlinear laser equation including loss terms is solved to find the time-evolution of the photon-distribution function. A Laplace transform method is used and the large number of terms are taken care of by suitable and reasonable simplification.

The relaxation parameter γ_{ab} and the Doppler parameter ku are incorporated into the equation for an inhomogeneously broadened laser following an earlier work of Mohanty and Nayak (1976) which avoids the Doppler-limit approximation.

The equation of motion for the photon-probability distribution function is given in Section 2. The time-dependent analytic solution of the equation neglecting the loss terms is derived in Section 3. A general time-dependent analytic solution is derived in Section 4. In Section 5 the results are discussed and the effects of loss and relaxation parameters on the time variation of the photon distribution function are illustrated graphically.

2. The non-linear laser equation

Riska and Stenholm (1970b) have given the non-linear equation for a single-photon inhomogeneously broadened detuned laser from quantum theory with

Doppler-limit approximation. Subsequently Mohanty and Nayak (1976) have lifted this restriction to extend the scope of the equation. The equation given by them is

$$\dot{\rho}_n = -A(n+1)\left\{ U(x,y) - \frac{n+1}{4}\frac{B}{A}\left[U(x,y) + \frac{y}{2}\left(\frac{\partial V(x,y)}{\partial x} - \frac{\partial U(x,y)}{\partial y}\right) + \frac{y^2}{y^2+x^2}\left(U(x,y) + \frac{y}{x}V(x,y)\right)\right]\right\}\rho_n + C(n+1)\rho_{n+1}$$

$$-\text{the same terms with } n \text{ replaced by } (n-1) \tag{1}$$

where

$$x = \Delta/ku, \qquad y = \gamma_{ab}/ku,$$

$$A = \frac{2\sqrt{\pi}g^2 r_a}{\gamma_a ku}, \qquad B = \frac{4g^2}{\gamma_a\gamma_b}A, \qquad C = \frac{2g^2 r_b}{\gamma_a\gamma_b},$$

g is the coupling constant, r_a and r_b are the rates of injection of atoms to a and b level respectively, γ_a, γ_b are the decay constants of the two levels, $\gamma_{ab} = \frac{1}{2}(\gamma_a+\gamma_b)$ and Δ is the detuning. $U(x, y)$ and $V(x, y)$ being the real and imaginary parts of the integral $\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{z-t}dt$, where $z = x+iy$ and whose values can be found from Faddeyeva and Terent'ov (1961). Equation (1) can be written as

$$\dot{\rho}_n = -\{A'-B'(n+1)\}(n+1)\rho_n + (A'-B'n)n\rho_{n-1} - Cn\rho_n + C(n+1)\rho_{n+1} \tag{2}$$

where

$$A' = AU(x, y)$$

$$B' = \frac{1}{4}B\left[U(x,y) + \frac{y}{2}\left(\frac{\partial V(x,y)}{\partial x} - \frac{\partial U(x,y)}{\partial y}\right) + \frac{y^2}{y^2+x^2}\left(U(x,y) + \frac{y}{x}V(x,y)\right)\right].$$

3. Solution of laser equation without loss terms

Equation (2) without the loss terms reduces to

$$\dot{\rho}_n = -\{A'-B'(n+1)\}(n+1)\rho_n + (A'-B'n)n\rho_{n-1}. \tag{3}$$

Taking the Laplace transform of the above equation and writing

$$\phi_n(s) = \int_0^\infty \rho_n(t)e^{-st}dt,$$

one obtains

$$s\phi_n(s) - \rho_n(0) = -[A'-B'(n+1)](n+1)\phi_n(s) + [A'-B'n]n\phi_{n-1}(s)$$

which gives

$$\phi_n(s) = \sum_{\alpha=0}^{n} \frac{\prod_{j=0}^{\alpha-1} [(n-j)\{A'-B'(n-j)\}]}{\prod_{j=0}^{\alpha} [s+(n-j+1)\{A'-B'(n-j+1)\}]} \rho_{n-\alpha}(0) \tag{4}$$

An inverse Laplace transform of the above gives

$$\rho_n(t) = \sum_{\alpha=0}^{n} \sum_{k=0}^{\alpha} \frac{\prod_{j=0}^{\alpha-1} [(n-j)\{A'-B'(n-j)\}]\exp[-(n+1-k)\{A'-B'(n+1-k)\}t]}{\prod_{\substack{j=0 \\ j\neq k}}^{\alpha} [(k-j)\{A'+B'(k+j-2n-2)\}]}$$

For initial vacuum state,

$$\rho_{n-\alpha}(0) = \begin{cases} 1 & \text{for } n = \alpha \\ 0 & \text{for } n \neq \alpha \end{cases}$$

Hence the above expression reduces to

$$\rho_n(t) = \sum_{k=0}^{n} \frac{\prod_{j=0}^{n-1} [(n-j)\{A'-B'(n-j)\}]\exp[-(n+1-k)\{A'-B'(n+1-k)\}t]}{\prod_{\substack{j=0 \\ j\neq k}}^{n} [(k-j)\{A'+B'(k+j-2n-2)\}]}. \tag{5}$$

4. General (time-dependent) solution of laser equation with loss terms

To obtain a general time-dependent solution of equation (2), Laplace transform method is made use of again. But here the main difference from previous works lies in the fact that both ascending and descending terms like ρ_{n+1} and ρ_{n-1} are present rendering the calculations relatively involved

A Laplace transform of equation (2) gives

$$\phi_n(s) = \frac{1}{[s+\{A'-B'(n+1)\}(n+1)+Cn]}$$

$$\times\{\rho_n(0)+(A'-B'n)n\phi_{n-1}(s)+C(n+1)\phi_{n+1}(s)\}. \tag{6}$$

Expansion of R.H.S. terms shows that the number of terms tends to increase indefinitely due to the third term. To take care of this a maximum limit is imposed on the photon number as N. Then the series, after some calculations, reduces to

$$
\begin{aligned}
\phi_n(s)[s+\{A'-B'(n+1)\}(n+1)+Cn] &= \rho_n(0) \\
&+ \sum_{\alpha=1}^{n} \prod_{j=0}^{\alpha-1} \frac{[A'-B'(n-j)](n-j)\,\rho_{n-\alpha}(0)}{[s+\{A'-B'(n-j)\}(n-j)+C(n-1-j)]} \\
&+ \sum_{\alpha=1}^{N} \prod_{j=0}^{\alpha-1} \frac{[C(n+1+j)]\,\rho_{n+\alpha}(0)}{[s+\{A'-B'(n+2+j)\}(n+2+j)+C(n+1+j)]} \\
&+\phi_n(s)\Bigg[\sum_{j=0}^{n-1} C^{j+1}[s+\{A'-B'(n-j)\}(n-j)+C(n-1-j)]\left(\frac{\underline{|n}}{\underline{|n-1-j}}\right)^2 \\
&\times \prod_{k=0}^{j} \frac{[A'-B'(n-k)]}{[s+\{A'-B'(n-k)\}(n-k)+C(n-1-k)]^2} \\
&+ \sum_{j=0}^{N} C^{j+1}[s+\{A'-B'(n+2+j)\}(n+2+j)+C(n+1+j)]\left(\frac{\underline{|n+j+1}}{\underline{|n}}\right)^2 \\
&\times \prod_{k=0}^{j} \frac{[A'-B'(n+1+j)]}{[s+\{A'-B'(n+k+2)\}(n+k+2)+C(n+1+k)]^2}.
\end{aligned} \tag{7}
$$

A close observation of the different terms of each of the above series reveals that they are convergent, the succeeding terms being much smaller compared to their previous ones. This justifies the retaining of only the first few terms. The condition of the initial state

$$
\rho_n(0) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}
$$

helps us to simplify the expression further. On the incorporation of these and with other necessary simplification equation (7) reduces to

$$
\begin{aligned}
\phi_n(s) = &\frac{\prod_{j=0}^{n-1} [A'-B'(n-j)](n-j)}{\prod_{j=0}^{n} [s+\{A'-B'(n+1-j)\}(n+1-j)+C(n-j)]} \\
&\times\Bigg[1+\frac{(A'-B'n)Cn^2}{[s+\{A'-B'n\}n+C(n-1)][s+\{A'-B'(n+1)\}(n+1)+Cn]} \\
&+\frac{\{A'-B'(n+1)\}C(n+1)^2}{[s+\{A'-B'(n+2)\}(n+2)+C(n+1)][s+\{A'-B'(n+1)\}(n+1)+Cn]}\Bigg]
\end{aligned} \tag{8}
$$

To obtain the inverse Laplace transform of equation (8), the Convolution theorem

$$L^{-1}[f(s)g(s)] = [L^{-1}f(s)]*[L^{-1}g(s)] = \int_0^t f(u)g(\tau-u)du$$

and standard tables of inverse transform (Erdelyi *et al* 1954) are made use of. Further it is seen that

$$\left[\frac{1}{\prod\limits_{j=0}^{n}[s+\{A'-B'(n+1-j)\}(n+1-j)+C(n-j)]}\right]$$

may contain double poles if the following conditions exist :

$$j \geqslant (n+1)-\frac{m+1}{2} \qquad \text{for odd } m$$

$$j \geqslant (n+1)-\frac{m}{2} \qquad \text{for even } m$$

where

$$m = \frac{A'+C}{B'} = \text{An integer.} \tag{9}$$

This case when condition (9) exists will be discussed later. First in the more general case when the above condition (9) does not exist inverse transform equation (8) gives

$$\rho_n(t) = F(n)\sum_{k=0}^{n}\frac{\exp.(-[\{A'-B'(n+1-k)\}(n+1-k)+C(n-k)]t)}{\prod\limits_{\substack{j=0\\j\neq k}}^{n}(k-j)\{A'+C'+B'(k+j-2n-2)\}}$$

$$+F(n)\frac{(A'-B'n)Cn^2}{\{A'+C-B'(2n+1)\}}\sum_{k=0}^{n}\frac{1}{\prod\limits_{\substack{j=0\\j\neq k}}^{n}(k-j)\{A'+C+B'(k+j-2n-2)\}}$$

$$\times\left[\exp[-\{(A'-B'n)n+C(n-1)\}t]\left\{\frac{1-\exp[-(1-k)\{A'+C-B'(2n+1-k)\}t]}{(1-k)\{A'+C-B'(2n+1-k)\}}\right\}\right.$$

$$\left.+\exp(-[\{A'-B'(n+1)\}(n+1)+Cn]t)\left\{\frac{1-\exp[k\{A'+C-B'(2n+2-k)\}t]}{k\{A'+C-B'(2n+2-k)\}}\right\}\right]+$$

$$F(n)\{A'-B'(n+1)\}C(n+1)^2\sum_{k=0}^{n+1}\frac{\exp(-[\{A'-B'(n+1)\}(n+1)+Cn]t)}{\prod\limits_{\substack{j=0\\j\neq k}}^{n+1}(k-j)\{A'+B'(k+j-2n-4)+C\}}$$

$$\times\left\{\frac{1-\exp[-(1-k)\{A'+C-B'(2n+3-k)\}t]}{(1-k)\{A'+C-B'(2n+3-k)\}}\right\}, \tag{10}$$

where

$$F(n) = \prod_{j=0}^{n-1} [A'-B'(n-j)](n-j).$$

5. Results and discussion

The expression for $\rho_n(t)$ in equation (5) is an exact one and the general result of equation (10), as expected, reduces to equation (5) when loss terms are absent ($C = 0$). Moreover, at initial stage for $t = 0$ the equation (10) gives

$$\rho_n(0) = 0 \quad \text{for } n \neq 0$$
$$= 1 \quad \text{for } n = 0$$

which further confirms the correctness of the result.

A' and B' are determined for various values of the relaxation parameters γ_{ab}/ku and Δ/ku from the tables of Faddeyeva and Terent'ev (1961) and the values of the probability distribution function at different time intervals are evaluated through a computer calculation.

Figure 1 shows the time-variation of $\rho_n(\tau)$, where $\tau = A't$, without loss corresponding to equation (5) and the time-variation of $\rho_n(\tau)$ with loss corresponding to equation (10) is shown in Figure 2 for different values of the relaxation parameters and for the specific case when $n = 5$.

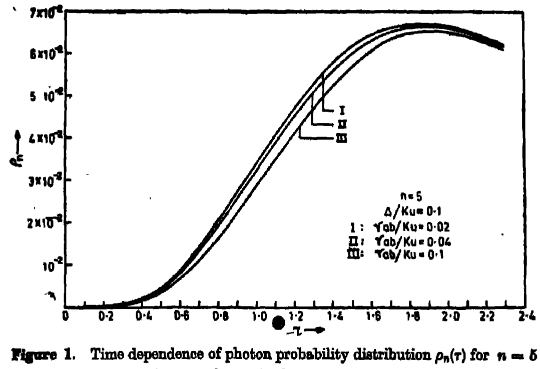

Figure 1. Time dependence of photon probability distribution $\rho_n(\tau)$ for $n = 5$ (without loss) at different values of relaxation parameters. ($B'/A' = 0{\cdot}008$).

The distribution function $\rho_n(\tau)$ is seen to build upto a peak value with increasing time in the absence of the loss (c) term (Figure 1). This peak, as well as the ρ_n values at different times are strongly dependent on γ_{ab}/ku and Δ/ku, the dependence being emphasized near the peak value itself. When loss terms are added (Figure 2) the qualitative conclusion is the same, but the peak, which is found lower than that without loss, is reached earlier and the influence of γ_{ab}/ku is stronger at the peak. The value of $\rho_n(\tau)$ decreases with increasing time and is expected to reach the steady state value after a long time.

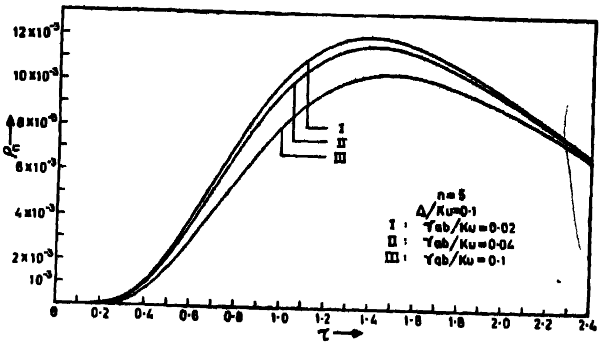

Figure 2. Time dependence of photon probability distribution $\rho_n(\tau)$ for $n = 5$ (with loss) at different values of relaxation parameters. ($B'/A' = 0{\cdot}003$, $C/A' = 0{\cdot}8$).

It is seen that the process of reaching the peak and approaching the steady state value is slowed down with increasing value of γ_{ab}/ku. This is intimately connected to the fact that at steady state the ρ_n vs. n curve will be broadened and the difference between various ρ_n's will be less for higher γ_{ab}/ku (Mohanty and Nayak 1976). That the influence of relaxation term is pronounced at the peak in presence of loss (c) is explained by the fact that at steady state the ρ_n which is maximum shows the most pronounced effect of the relaxation term γ_{ab}/ku. This is required to flatten the ρ_n vs. n curve with increasing γ_{ab}/ku.

In this paper we have dealt with only one ρ_n as we want to study the effect of time evolution on a typical member of the family of distribution functions. The value of $n = 5$, chosen at random, shows a peak at $\tau = 1{\cdot}4$ (with loss term) going down to its steady state value which is less than the peak value. $\rho_0(\tau)$, as expected, peaks at $\tau = 0$ (equation 10). All these terms will decrease with increasing τ due to the presence of the exponential terms in the numerators of equation (10). The only term which will peak (or keep increasing as τ increases) will be the one whose denominators are also low. All $\rho_n(\tau)$ seem to decrease with τ, but this needs cause no concern as all ρ_n terms are unnormalized at present and hence when normalized (i.e., $\sum_n \rho_n = 1$ for all τ) the relative and not the actual magnitudes of $\rho_n(\tau)$ will matter. This should be done computorially as the difficulties in finding the maximum ρ_n with respect to n analytically even at steady state is well known. (Riska and Stenholm 1970a, 1970b, Mohanty and Nayak 1976).

We would now discuss the method for including the double pole in the Laplace transformed ρ_n in equation (8), whihch is done as follows. When condition (9) exists then to obtain the inverse transform of (8) the quantity

$$\frac{1}{\prod_{j=0}^{n}[s+\{A'-B'(n+1-j)\}(n+1-j)+C(n-j)]}$$

is written as

$$\frac{1}{\prod_{j=0}^{l}[s+\{A'-B'(n+1-j)\}(n+1-j)+C(n-j)]}$$

$$\frac{1}{\prod_{j'=l+1}^{n}[s+\{A'-B'(n+1-j')\}(n+1-j')+C(n-j')]}$$

where

$$\begin{aligned} l &= p/2 && \text{for even } p \\ &= \frac{p-1}{2} && \text{for odd } p \qquad p = 2(n+1)-m \end{aligned}$$

and the inverse transform obtained by repeated application of convolution theorem is given by

$$\rho_n(t) = F(n)\sum_{k=0}^{l}\sum_{k'=l+1}^{n}\frac{1}{\prod_{\substack{j=0\\ j\neq k}}^{l}(f_j-f_k)\prod_{\substack{j'=l+1\\ j'\neq k'}}^{n}(f_{j'}-f_{k'})}$$

$$\times\delta_{f_k f_{k'}}\, t\exp(-f_{k'}t)+(1-\delta_{f_k f_{k'}})\left\{\frac{\exp(-f_k t)-\exp(-f_{k'}t)}{(f_k-f_{k'})}\right\}\Bigg]$$

$$+\frac{F(n)\{(A'-B'n)Cn^2\}}{\{A'+C-B'(2n+1)\}}\sum_{k=0}^{l}\sum_{k'=l+1}^{n}\frac{1}{\prod_{\substack{j=0\\ j\neq k}}^{l}(f_j-f_k)\prod_{\substack{j'=l+1\\ j'\neq k'}}^{n}(f_{j'}-f_{k'})}$$

$$\times\Big[\delta_{f_k f_{k'}}\frac{\exp(-f_1t)}{(f_{k'}-f_1)^2}\{1-(f_{k'}-f_1)t\exp\{-(f_{k'}-f_1)t\}-\exp(\{-(f_{k'}-f_1)t\}\}$$

$$-\delta_{f_k f_{k'}}\frac{\exp(-f_0t)}{(f_{k'}-f_0)^2}\{1-(f_{k'}-f_0)t\exp\{-(f_{k'}-f_0)t\}-\exp\{-(f_{k'}-f_0)t\}\}$$

$$+(1-\delta_{f_k f_{k'}})\frac{1}{(f_k-f_{k'})}\Big\{\frac{\exp(-f_1t)-\exp(-f_kt)}{f_k-f_1}-\frac{\exp(-f_0t)-\exp(-f_kt)}{f_k-f_0}$$

$$-\frac{\exp(-f_1t)-\exp(-f_{k'}t)}{f_{k'}-f_1}+\frac{\exp(-f_0t)-\exp(-f_{k'}t)}{f_{k'}-f_0}\Big\}\Big]$$

$$+F(n)\frac{\{A'-B'(n+1)\}C(n+1)^2}{\{A'+C-B'(2n+3)\}}\sum_{k=0}^{l}\sum_{k'=l+1}^{n}\frac{1}{\prod\limits_{\substack{j=0\\j\neq k}}^{l}(f_j-f_k)\prod\limits_{\substack{j'=l+1\\j'\neq k'}}^{n}(f_{j'}-f_{k'})}$$

$\times$[Terms inside previous square bracket with

f_1 replaced by f_0 and f_0 by f_{-1}],

where

$$f_j=[\{A'-B'(n+1-j)\}(n+1-j)+C(n-j)]$$

$$\delta_{f_k f_{k'}}=1 \qquad \text{for } k=k'$$

$$=0 \qquad \text{for } k\neq k'.$$

One of the highlights of this paper is the conclusion that one can get a viable equation of motion and a solution to it when both the ascending and descending terms of $\rho_n(\tau)$ are occurring along with the non-linear parameters in the equation of motion for ρ_n. The result of this conclusion is the evolution of a technique which can be used in other problems of non-linear optics. This has been done, as an example, for Stimulated Raman Scattering, the report of which is presented elsewhere in this conference.

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